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Improved Orbit Estimation by Satellite-Mounted Gravity Gradiometers

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The feasibility of the use of gravity gradiometers mounted on an Earth satellite in order to improve the estimation of its orbit is studied. Only the high-frequency orbit residuals that are caused by the uncertainty in our knowledge of the Earth's field are considered. Using the principle of minimum variance filtering, critical values of the variance of the mean instrument error, $\sigma_\Gamma^2\tau/kT$, are calculated at different altitudes under the assumption that harmonics beyond the 8×8 field are unknown (σ_Γ^2 = instrument error variance, τ = instrument time constant, T = satellite period, kT = duration of filter window). It is found that instruments presently in development may lead to significantly improved ephemerides, especially at low altitudes.

Introduction

IN this study we investigate the feasibility of using gravity gradiometers mounted on an Earth satellite to improve the estimation of its orbit. Typically, an orbit is determined by combining observational data (e.g., ranges or range rates) with a dynamic model based on the known harmonics in the Earth's field. Even when many observations are available, an irreducible uncertainty remains due to our imperfect knowledge of the Earth's field. Can this residual uncertainty be reduced by utilizing onboard gravity gradiometers? A number of authors have considered terrestrial inertial navigators assisted by moving base gravity gradiometers to correct for imperfect gravity compensation.[1-4] The situation for a satellite is reversed; the principal source of its orbit uncertainty is due to imperfect knowledge of the gravity field. The primary navigation device might be a set of gradiometers, assisted by accelerometers to compensate for nongravitational accelerations.

There are several moving-base gradiometers currently in development.[4-7] Let us assume a nearly circular orbit and data over one revolution. For such a short arc we may neglect the rotation of the Earth and the precession of the orbit. We further assume that the Earth's field is known perfectly up to and including harmonics of degree $L-1$. We call this "the modeled field." To this is added "the unmodeled field" consisting of all harmonics of degree greater or equal to L. The unmodeled field is regarded as stochastic in nature, its statistics determined by its harmonic coefficients. We assume that their expected values vanish, that they are uncorrelated, and that their variances obey Kaula's attenuation rule (Ref. 8, Chap. 3). The validity of this attenuation rule appears to be confirmed up to 75 deg.[9] We shall be concerned solely with the reduction of the high-frequency orbit errors. In view of the assumptions made about the unmodeled field, which imply homogeneity and isotropy (Ref. 10, Sec. 17), we may consider an equatorial orbit.

Regarding the nominal orbit as that determined from the tracking data and the modeled field, the instrument readings differ from their nominal values within linear approximation because of 1) the unmodeled gravity field at the nominal orbit, 2) the displacement from the nominal orbit due to the unmodeled field, and 3) instrument errors. Clearly the perturbations from the nominal orbit are correlated with the contributions of 1 and 2 to the gradiometer output.

We assume the instrument error to be represented by white noise with the power spectral density of $\sigma_\Gamma^2\tau$ where σ_Γ^2 is the error variance and τ is the integration time. Instrument bias and drift are neglected. They are of lesser importance here than for other applications, because we are concerned solely with periodic orbit perturbations.

It is natural to apply the method of minimum variance estimation to improve the orbit (Ref. 11, Chap. 11). We shall use bold type to indicate stochastic quantities in what follows.

The Stochastic Model of the Signal and the Data

The unmodeled part of the Earth's gravity potential may be written

$$V=\sum_{l=L}^{\infty}\sum_{m=-l}^{l}\frac{\mu R^l}{r^{l+1}}C(l,m)\,U(l,m;\varphi,\lambda)\equiv\sum_{l=L}^{\infty}\sum_{m=-l}^{l}V_{lm} \qquad (1)$$

where

r = radius
ϕ = latitude
λ = longitude
R = mean radius of Earth (6371 km)
μ = gR^2
$C(l,m)$ = harmonic coefficients regarded as random variables

The normed complex harmonics are defined by

$$U(l,m;\lambda,\varphi)=N(l,m)\,P_l^{|m|}(\sin\varphi)\,e^{im\lambda} \qquad (2a)$$

$$N(l,m)=\sqrt{(2l+1)\,\frac{(l-|m|)!}{(l+|m|)!}} \qquad (2b)$$

The harmonic coefficients satisfy

$$C(l,m)^*=C(l,-m) \qquad (3a)$$

$$E\{C(l,m)\}=0 \qquad (3b)$$

$$E\{C(l,m)\,C(l',m')^*\}=10^{-10}l^{-4}\delta_{l,l'}\delta_{m,m'} \qquad (3c)$$

As a stochastic model for the high-frequency part of the orbit error we take the solution of the linearized variational

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equations for circular orbit. They are

$$\delta\ddot{\rho}-3n^2\delta\rho-2an\delta\dot{u}=\left(\frac{\partial V}{\partial r}\right) \tag{4a}$$

$$a\delta\ddot{u}+2n\delta\dot{\rho}=\frac{1}{a}\left(\frac{\partial V}{\partial\lambda}\right) \tag{4b}$$

$$\delta\ddot{\zeta}+n^2\delta\zeta=\frac{1}{a}\left(\frac{\partial V}{\partial\varphi}\right) \tag{4c}$$

the partial derivatives of V are evaluated at $r=a$, $\lambda=nt$, $\phi=0$, where

- a = orbit radius
- n = mean motion
- $\delta\rho$ = radial orbit perturbation
- $a\delta u$ = along-track orbit perturbation
- $\delta\zeta$ = out-of-plane orbit perturbation

Based on the definitions of the Legendre polynomials and the associated functions of Ferrers[12] one may show, using Stirling's approximation for the factorials, that near the equatorial plane:

For $l-m$ even and $l>m$

$$V_{lm}\approx\frac{\mu R^l}{r^{l+1}}C(l,m)(-1)^{(l-m)/2}\frac{2}{\sqrt{\pi}}\cdot\frac{(1+1/2l)^{1/2}}{(1-m^2/l^2)^{1/4}}\{1-\tfrac{1}{2}[(l+m+1)(l-m)+m]\sin^2\varphi\}e^{im\lambda} \tag{5}$$

For $l=m$

$$V_{mm}\approx\frac{\mu R^m}{r^{m+1}}C(m,m)\frac{\sqrt{2}}{\pi^{1/4}}m^{1/4}\left(1+\frac{1}{2m}\right)^{1/2}\left\{1-\tfrac{1}{2}m\sin^2\varphi\right\}e^{im\lambda} \tag{6}$$

For $l-m$ odd

$$V_{lm}\approx\frac{\mu R^l}{r^{l+1}}C(l,m)(-1)^{(l-m-1)/2}\frac{2l}{\sqrt{\pi}}\times\left[\left(1+\frac{1}{l}\right)\left(1+\frac{1}{2l}\right)\right]^{1/2}\left[1-\frac{m^2}{(l+1)^2}\right]^{1/4}\sin\varphi\, e^{im\lambda} \tag{7}$$

The errors in these expressions are less than 10% in consequence of the Stirling approximation.

From the form of the generic potential terms V_{lm} one observes that the in-plane perturbations are excited by the $l-m$ even terms and the out-of-plane perturbations by the $l-m$ odd terms. We confine ourselves to the study of the in-plane perturbations $\delta\rho$ and $a\delta u$. (The radial perturbations $\delta\rho$ is the principal contributor to the error in geoid determination by satellite altimetry.) We deal only with those frequencies for which $m\geq 2$. The solution of Eqs. (4) for one generic potential term for $l-m$ even and $l>m$ is

$$\delta\rho_{lm}=(-1)^{(l-m)/2}\frac{2a}{\sqrt{\pi}}\left(\frac{R}{a}\right)^l C(l,m)\cdot\frac{l-1}{m^2-1}\cdot\left(1+\frac{1}{2l}\right)^{1/2}\left(1-\frac{m^2}{l^2}\right)^{-1/4}e^{imnt} \tag{8}$$

$$a\delta u_{lm}=i(-1)^{(l-m)/2}\frac{2a}{\sqrt{\pi}}\cdot\left(\frac{R}{a}\right)^l C(l,m)\frac{2l-1-m^2}{m(m^2-1)}\times\left(1+\frac{1}{2l}\right)^{1/2}\left(1-\frac{m^2}{l^2}\right)^{-1/4}e^{imnt} \tag{9}$$

and for $l=m$

$$\delta\rho_{mm}=\frac{\sqrt{2}a}{\pi^{1/4}}\left(\frac{R}{a}\right)^m C(m,m)\cdot\frac{m^{1/4}}{m+1}\cdot\left(1+\frac{1}{2m}\right)^{1/2}e^{imnt} \tag{10}$$

$$a\delta u_{mm}=-i\frac{\sqrt{2}a}{\pi^{1/4}}\left(\frac{R}{a}\right)^m C(m,m)\cdot\frac{m-1}{m+1}m^{-1/4}\left(1+\frac{1}{2m}\right)^{1/2}e^{imnt} \tag{11}$$

To each term we must consider added its complex conjugate. To obtain the total orbit error of frequency mn one must sum over l so that $l-m$ is even and $l\geq\max.(L,m)$. In view of the assumptions about $C(l,m)$ [see Eq. (3)], the stochastic processes $\delta\rho$ and $a\delta u$ are stationary and periodic in this simplified model (Ref. 11, p. 368).

To simplify our study further, let us assume that we have two instruments yielding the combination

$$-\tfrac{1}{2}(\Gamma_{yy}-\Gamma_{zz})+\tfrac{1}{2}(\Gamma_{zz}-\Gamma_{xx})=3\Gamma_{zz}/2 \tag{12}$$

at the center of mass of the vehicle. We shall use this quantity alone to estimate the in-plane perturbations. Naturally, this does not lead to the best estimator. Nevertheless, we shall be able to determine the required instrument precision.

From Eqs. (5-7) one obtains for the contribution to $3\Gamma_{zz}/2$ due to the unmodeled field (1 in the Introduction) the following generic terms

$$\frac{3}{2}\frac{\partial^2}{\partial z^2}V_{lm}=-(-1)^{(l-m)/2}\frac{3n^2}{\sqrt{\pi}}\left(\frac{R}{a}\right)^l C(l,m)[(l+1)^2-m^2]\times\left(1+\frac{1}{2l}\right)^{1/2}\left(1-\frac{m^2}{l^2}\right)^{-1/4}e^{imnt} \tag{13}$$

for $l-m$ even and $l>m$, and

$$\frac{3}{2}\frac{\partial^2}{\partial z^2}V_{mm}=-\frac{3n^2}{\sqrt{2}\pi^{1/4}}\left(\frac{R}{a}\right)^m C(m,m)m^{1/4}(2m+1)\left(1+\frac{1}{2m}\right)^{1/2}e^{imnt} \tag{14}$$

for $l=m$.

To these expressions must be added the contribution due to the displacement (2 in the Introduction). The central term in the Earth's field being about 1000 times the leading correction term, it suffices to consider it alone. The contribution due to displacement from nominal orbit is

$$\frac{\partial}{\partial r}\left(\frac{3}{2}\frac{\partial^2}{\partial z^2}\frac{\mu}{r}\right)\delta\rho=\frac{9}{2}\frac{n^2}{a}\delta\rho \tag{15}$$

For a generic potential term for which $l-m$ is even and $l>m$, this becomes

$$\frac{9}{2}\frac{n^2}{a}\delta\rho_{lm}=(-1)^{(l-m)/2}\frac{9n^2}{\sqrt{\pi}}\left(\frac{R}{a}\right)^l C(l,m)\frac{l-1}{m^2-1}\times\left(1+\frac{1}{2l}\right)^{1/2}\left(1-\frac{m^2}{l^2}\right)^{-1/4}e^{imnt} \tag{16}$$

and for $l=m$

$$\frac{9}{2}\frac{n^2}{a}\delta\rho_{mm}=\frac{9n^2}{\sqrt{2}\pi^{1/4}}\left(\frac{R}{a}\right)^m C(m,m)\frac{m^{1/4}}{m+1}\left(1+\frac{1}{2m}\right)^{1/2}e^{imnt} \tag{17}$$

Adding the contributions due to 1 and 2 one obtains for $l-m$ even and $l>m$

$$\left(\frac{3}{2}\Gamma_{zz}\right)_{lm}\equiv z_{lm}=-(-1)^{(l-m)/2}\frac{3n^2}{\sqrt{\pi}}\left(\frac{R}{a}\right)^l C(l,m)\times\left[(l+1)^2-m^2-\frac{3(l-1)}{m^2-1}\right]\left(1+\frac{1}{2l}\right)^{1/2}\left(1-\frac{m^2}{l^2}\right)^{-1/4}e^{imnt} \tag{18}$$

and for $l=m$

$$\left(\frac{3}{2}\Gamma_{zz}\right)_{mm} \equiv z_{mm} = -\frac{3n^2}{\sqrt{2}\pi^{1/4}}\left(\frac{R}{a}\right)^m C(m,m)m^{1/4} \times\left(2m+1-\frac{3}{m+1}\right)\left(1+\frac{1}{2m}\right)^{1/2} e^{im\,nt} \tag{19}$$

One may now apply the principle of minimum variance to estimate $\delta\rho$ and $a\delta u$ from z + noise. (It has been brought to the author's attention by a reviewer, that Glaser[13] has derived general relations between the gravity gradients anywhere aloft and the spherical harmonic expansion of the Earth's field.)

The Filter

Applying the principle of minimum variance estimation (Ref. 11, Chap. 11) one needs the expected powers and cross powers of ($\delta\rho$, $a\delta u$) and z at the frequencies mn. In the summations that follow, l extends to infinity and so that $l-m$ is even. The lower limits of summation are given by these rules: 1) for $m<L$ take the least l such that $l\geq L$ and $l-m$ is even, and 2) for $m\geq L$ take $l=m+2$. The expected powers are denoted by P and the cross powers by C. The formulas follow:

For $m<L$

$$P(\delta\rho;\,m)=\frac{4a^2\times10^{-10}}{\pi(m^2-1)^2}\sum_l\left[\left(\frac{R}{a}\right)^{2l}\frac{1}{l^2}\left(1-\frac{1}{l}\right)^2\left(1+\frac{1}{2l}\right)\left(1-\frac{m^2}{l^2}\right)^{-1/2}\right] \tag{20}$$

For $m\geq L$

$$P(\delta\rho;\,m)=\frac{2a^2\times10^{-10}}{\sqrt{\pi}}\left(\frac{R}{a}\right)^{2m}m^{-11/2}\left(1+\frac{1}{m}\right)^{-2}\left(1+\frac{1}{2m}\right)+\frac{4a^2\times10^{-10}}{\pi(m^2-1)^2}\sum_l\left[\left(\frac{R}{a}\right)^{2l}\frac{1}{l^2}\left(1-\frac{1}{l}\right)^2\left(1+\frac{1}{2l}\right)\left(1-\frac{m^2}{l^2}\right)^{-1/2}\right] \tag{21}$$

For $m<L$

$$P(a\delta u;\,m)=\frac{16a^2\times10^{-10}}{\pi m^2(m^2-1)^2}\sum_l\left[\left(\frac{R}{a}\right)^{2l}\frac{1}{l^2}\left(1-\frac{m^2+1}{2l}\right)^2\left(1+\frac{1}{2l}\right)\left(1-\frac{m^2}{l^2}\right)^{-1/2}\right] \tag{22}$$

For $m\geq L$

$$P(a\delta u;\,m)=\frac{2a^2\times10^{-10}}{\sqrt{\pi}}\left(\frac{R}{a}\right)^{2m}m^{-11/2}\left(\frac{m-1}{m+1}\right)^2\left(1+\frac{1}{2m}\right)+\frac{16a^2\times10^{-10}}{\pi m^2(m^2-1)^2}\sum_l\left[\left(\frac{R}{a}\right)^{2l}\frac{1}{l^2}\times\left(1-\frac{m^2+1}{2l}\right)^2\left(1+\frac{1}{2l}\right)\left(1-\frac{m^2}{l^2}\right)^{-1/2}\right] \tag{23}$$

For $m<L$

$$P(z;\,m)=\frac{9n^4\times10^{-10}}{\pi}\sum_l\left[\left(\frac{R}{a}\right)^{2l}\left\{\left(1+\frac{1}{l}\right)^2-\left(\frac{m}{l}\right)^2-\frac{3}{l}\left(1-\frac{1}{l}\right)\frac{1}{m^2-1}\right\}^2\left(1+\frac{1}{2l}\right)\left(1-\frac{m^2}{l^2}\right)^{-1/2}\right] \tag{24}$$

For $m\geq L$

$$P(z;\,m)=\frac{18n^4\times10^{-10}}{\sqrt{\pi}}\left(\frac{R}{a}\right)^{2m}m^{-3/2}\left[1+\frac{1}{2m}-\frac{3}{2m(m+1)}\right]^2\left(1+\frac{1}{2m}\right)+\frac{9n^4\times10^{-10}}{\pi}\sum_l\left[\left(\frac{R}{a}\right)^{2l}\left\{\left(1+\frac{1}{l}\right)^2-\left(\frac{m}{l}\right)^2-\frac{3}{l}\left(1-\frac{1}{l}\right)\frac{1}{m^2-1}\right\}^2\left(1+\frac{1}{2l}\right)\left(1-\frac{m^2}{l^2}\right)^{-1/2}\right] \tag{25}$$

In Eqs. (24) and (25) the terms involving $(m^2-1)^{-1}$ and $(m+1)^{-1}$ represent contribution 2 to the instrument output. They are due to the unmodeled displacement from the nominal orbit and contribute little, unless the modeled field is of low degree.

For $m<L$

$$C(z,\delta\rho;m)=-\frac{6an^2\times10^{-10}}{\pi(m^2-1)}\sum_l\left[\left(\frac{R}{a}\right)^{2l}\frac{1}{l}\left(1-\frac{1}{l}\right)\left\{\left(1+\frac{1}{l}\right)^2-\left(\frac{m}{l}\right)^2-\frac{3}{l}\left(1-\frac{1}{l}\right)\frac{1}{m^2-1}\right\}\left(1+\frac{1}{2l}\right)\left(1-\frac{m^2}{l^2}\right)^{-1/2}\right] \tag{26}$$

For $m\geq L$

$$C(z,\delta\rho;\,m)=-\frac{6an^2\times10^{-10}}{\sqrt{\pi}}\left(\frac{R}{a}\right)^{2m}\frac{m^{-5/2}}{m+1}\left(1+\frac{1}{2m}-\frac{3}{2m(m+1)}\right)\left(1+\frac{1}{2m}\right)-\frac{6an^2\times10^{-10}}{\pi(m^2-1)}\sum_l\left[\left(\frac{R}{a}\right)^{2l}\frac{1}{l}\left(1-\frac{1}{l}\right)\left\{\left(1+\frac{1}{l}\right)^2-\left(\frac{m}{l}\right)^2-\frac{3}{l}\left(1-\frac{1}{l}\right)\frac{1}{m^2-1}\right\}\left(1+\frac{1}{2l}\right)\left(1-\frac{m^2}{l^2}\right)^{-1/2}\right] \tag{27}$$

For $m<L$,

$$C(z,a\delta u;\ m) = -i\frac{12an^2\times 10^{-10}}{\pi m(m^2-1)}\sum_l\Big[\Big(\frac{R}{a}\Big)^{2l}\frac{1}{l}\Big(1-\frac{m^2+1}{2l}\Big)$$

$$\times\Big\{\Big(1+\frac{1}{l}\Big)^2-\Big(\frac{m}{l}\Big)^2-\frac{3}{l}\Big(1-\frac{1}{l}\Big)\frac{1}{m^2-1}\Big\}\Big(1+\frac{1}{2l}\Big)\Big(1-\frac{m^2}{l^2}\Big)^{-1/2}\Big] \tag{28}$$

For $m\geq L$

$$C(z,a\delta u;\ m) = i\frac{6an^2\times 10^{-10}}{\sqrt{\pi}}\Big(\frac{R}{a}\Big)^{2m}m^{-7/2}\frac{m-1}{m+1}\cdot\Big(1+\frac{1}{2m}-\frac{3}{2m(m+1)}\Big)\Big(1+\frac{1}{2m}\Big)$$

$$-i\frac{12an^2\times 10^{-10}}{\pi m(m^2-1)}\sum_l\Big[\Big(\frac{R}{a}\Big)^{2l}\frac{1}{l}\Big(1-\frac{m^2+1}{2l}\Big)\Big\{\Big(1+\frac{1}{l}\Big)^2-\Big(\frac{m}{l}\Big)^2-\frac{3}{l}\Big(1-\frac{1}{l}\Big)\frac{1}{m^2-1}\Big\}\Big(1+\frac{1}{2l}\Big)\Big(1-\frac{m^2}{l^2}\Big)^{-1/2}\Big] \tag{29}$$

The instrument output is assumed to be infected by white noise of power spectral density $\sigma_\Gamma^2\tau$. This leads by the standard methods (Ref. 11, Sec. 11.4) to the following formula for the variance of the estimator error at frequency mn:

$$\sigma^2(\delta\hat{\rho};\ m) = 2\Big[P(\delta\rho;\ m)-\frac{|C(z,\delta\rho;\ m)|^2}{P(z;\ m)+(\sigma_\Gamma^2\tau/kT)}\Big] \tag{30}$$

where

$T\ = 2\pi/n =$ satellite period
$kT\ =$ window duration of filter

A similar formula, with $P(a\delta u;\ m)$ replacing $P(\delta\rho;\ m)$ and $C(z,a\delta u;\ m)$ replacing $C(z,\delta\rho;\ m)$, holds for the estimation of $a\delta u$.

Numerical Results

We consider the Earth's harmonics up to and including the 8th degree to be well known. In Tables 1-4 are listed the expected powers and cross powers as given by Eqs. (20-29) for frequencies mn ($m=2,\ldots,8$) and for satellite altitudes of $H=150$, 300, 500, and 1000 km. The cutoff parameter has been set at $L=9$, according to our assumption that the 8×8 field is well known. It may be noted (see Table 2 for $H=300$ km) that the expected powers $P(z;\ m)$ are in order of magnitude in agreement with those obtained by Chovitz et al.[9] Inspection of Eq. (30) shows that the signal-to-noise ratio $\sigma_\Gamma^2\tau/kTP(z;\ m)$ must be less than unity in order for the satellite's ephemeris residuals to be appreciably reduced. We see from Tables 1-4 that at the low frequencies ($m=2,\ldots,8$), which correspond to the large perturbations, the gradient

Table 1 Powers and cross powers for $L=9$, $H=150$ km ($T=87.384$ min, $n=0.11984\times 10^2$ s^{-1})[a]

m	$P(\delta\rho;\ m)\times 10^{-5}$, cm^2	$P(z;\ m)\times 10^{20}$, s^{-4}	$C(z,\delta\rho;\ m)\times 10^8$, cm $\times$ s^{-2}	$P(a\delta u;\ m)\times 10^{-5}$, cm^2	$-iC(z,a\delta u;m)\times 10^8$, cm $\times$ s^{-2}
2	1.121996	0.466437	2.048239	0.874690	1.841949
3	0.194461	0.508662	0.882503	0.037603	0.413392
4	0.046619	0.475476	0.418503	0.002410	0.103779
5	0.023175	0.481628	0.288584	0.0005578	0.013287
6	0.009233	0.441572	0.173992	0.0004066	−0.009533
7	0.006731	0.430543	0.137487	0.0011358	−0.029120
8	0.003319	0.393054	0.091797	0.0006968	−0.024364

[a] $L-1=$ highest degree in the modeled field; $H=$ satellite altitude in km; $T=$ satellite period in min; $n=$ satellite mean motion in s^{-1}; $P(\delta\rho;\ m)=$ expected power of adial perturbation in cm^2 at frequency mn, as defined in Eqs. (20) and (21); $P(z;\ m)=$ expected power of gradient in s^{-4} at frequency mn, as defined in Eqs. (24) and (25); $C(z,\delta\rho;\ m)=$ expected cross power of gradient and radial perturbation in cm. s^{-2} at frequency mn, as defined in Eqs. (26) and (27); P($a\delta u;\ m$) = expected power of along-track perturbation in cm^2 at frequency mn, as defined in Eqs. (22) and (23); $C(z,\ a\delta u;\ m)=$ expected cross power of gradient and along-track perturbation in cm s^{-2} at frequency mn, as defined in Eqs. (28) and (29)

Table 2 Powers and cross powers for $L=9$, $H=300$ km ($T=90.418$ min, $n=0.11582\times 10^{-2}$ s^{-1})[a]

m	$P(\delta\rho;\ m)\times 10^{-5}$, cm^2	$P(z;\ m)\times 10^{20}$, s^{-4}	$C(z,\delta\rho;\ m)\times 10^8$, cm $\times$ s^{-2}	$P(a\delta u;\ m)\times 10^{-5}$, cm^2	$-iC(z,a\delta u;m)\times 10^8$, cm $\times$ s^{-2}
2	0.591875	0.140917	0.860890	0.449836	0.759778
3	0.108965	0.162027	0.393484	0.018931	0.171219
4	0.024728	0.142914	0.176085	0.0009647	0.036330
5	0.013117	0.148517	0.127909	0.0002948	−0.001821
6	0.004952	0.127589	0.072621	0.0002509	−0.008726
7	0.003894	0.124500	0.060195	0.0007606	−0.018003
8	0.001822	0.106229	0.037806	0.0004422	−0.013418

[a] $L-1=$ highest degree in the modeled field; $H=$ satellite altitude in km; $T=$ satellite period in min; $n=$ satellite mean motion in s^{-1}; $P(\delta\rho;\ m)=$ expected power of adial perturbation in cm^2 at frequency mn, as defined in Eqs. (20) and (21); $P(z;\ m)=$ expected power of gradient in s^{-4} at frequency mn, as defined in Eqs. (24) and (25); $C(z,\delta\rho;\ m)=$ expected cross power of gradient and radial perturbation in cm. s^{-2} at frequency mn, as defined in Eqs. (26) and (27); P($a\delta u;\ m$) = expected power of along-track perturbation in cm^2 at frequency mn, as defined in Eqs. (22) and (23); $C(z,\ a\delta u;\ m)=$ expected cross power of gradient and along-track perturbation in cm s^{-2} at frequency mn, as defined in Eqs. (28) and (29)

Table 3 Powers and cross powers for $L = 9$, $H = 500$ km ($T = 94.517$ min, $n = 0.11079 \times 10^{2} s^{-1}$)[a]

m	$P(\delta\rho;\, m) \times 10^{-5}$, cm^2	$P(z;\, m) \times 10^{20}$, s^{-4}	$C(z,\delta\rho;\, m) \times 10^{8}$, $cm \times s^{-2}$	$P(a\delta u;\, m) \times 10^{-5}$, cm^2	$-iC(z,a\delta u;m) \times 10^{8}$, $cm \times s^{-2}$
2	0.285629	0.0430607	0.338798	0.212727	0.294850
3	0.056448	0.0527333	0.165866	0.008958	0.068070
4	0.011989	0.0433847	0.069318	0.0003604	0.012181
5	0.006855	0.0469916	0.053653	0.0001590	−0.003177
6	0.002424	0.0374487	0.028403	0.0001410	−0.004782
7	0.002076	0.0371548	0.025002	0.0004562	−0.009118
8	0.0009104	0.0293290	0.014629	0.0002475	−0.006165

[a] Legend same as for Table 1.

Table 4 Powers and cross powers for $L = 9$, $H = 1000$ km ($T = 105.026$ min, $n = 0.99709 \times 10^{-3} s^{-1}$)[a]

m	$P(\delta\rho;\, m) \times 10^{-5}$, cm^2	$P(z;\, m) \times 10^{20}$, s^{-4}	$C(z,\delta\rho;\, m) \times 10^{8}$, $cm \times s^{-2}$	$P(a\delta u;\, m) \times 10^{-5}$, cm^2	$-iC(z,a\delta u;m) \times 10^{8}$, $cm \times s^{-2}$
2	0.0609565	0.00419152	0.0497992	0.0442052	0.0425944
3	0.0140805	0.00592580	0.0283792	0.0019683	0.0107762
4	0.0025757	0.00417734	0.0101871	0.00005108	0.0014097
5	0.0017318	0.00507776	0.0091191	0.00004655	−0.0010501
6	0.0005281	0.00344305	0.0041386	0.00003749	−0.0009414
7	0.0005400	0.00368260	0.0041956	0.00013837	−0.0018743
8	0.0002045	0.00246201	0.0021022	0.00006415	−0.0010584

[a] Legend same as for Table 1.

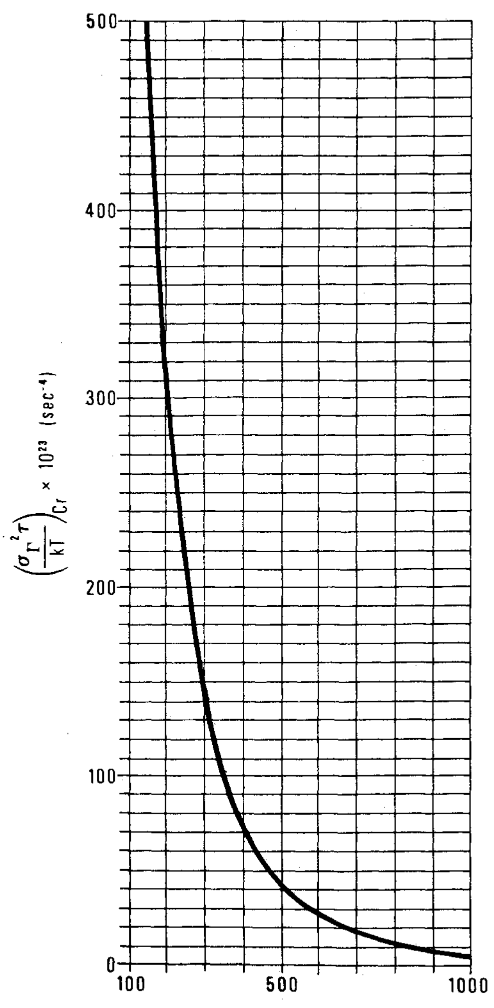

Fig. 1 Critical value of the instrument noise number vs satellite altitude.

power changes little with frequency. This value may be considered critical for $\sigma_\Gamma^2 \tau / kT$. In Fig. 1 the critial value of this noise number is plotted against the satellite's altitude. As an example take σ_Γ = 1Eötvös Unit (EU) = $10^{-9}\ s^{-2}$, $\tau = 10$ s. These constants correspond to presently proposed terrestrial instruments.

Consider a satellite at altitude $H = 300$ km and a data window of one revolution, $k = 1$, leading to $\sigma_\Gamma^2 \tau / T = 0.18 \times 10^{-20} s^{-4}$. This does not differ much from the critical value of $0.15 \times 10^{-20}\ s^{-4}$. It follows that at $H = 300$ km and with an instrument with a sensitivity of say $\sigma_\Gamma \approx 0.5$ EU and integration time of $\tau = 10$ s, a substantial orbit improvement is obtained. This is so, even though the signal amplitudes are much smaller than σ_Γ, $\sqrt{P(z;m)} \approx 0.04$ EU. In essence, one can get by with an instrument of lower precision than in Ref. 9 ($\sigma_\Gamma = 0.01$ EU, $\tau = 30$ s). The reason is that the output is sampled $540 = T/\tau$ times in one revolution to extract that part of the information which is relevant to the estimation of the orbit perturbations.

Substitution of $\sigma_\Gamma = 0.5$ EU, $\tau = 10$ s into Eq. (30) at an altitude of $H = 300$ km yields a variance reduction of 67%. The variance reduction would be greater if the optimal instrument output combination had been used.

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